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Symmetric closed characteristics on symmetric compact convex hypersurfaces in $\mathbf{R}^{2n}$

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In this article, let $\Sigma \subset \mathbf{R}^{2n}$ be a compact convex hypersurface which is symmetric with respect to the origin, where $n = 2$ or 3. We prove that if Σ carries exactly n geometrically distinct closed characteristics, then all of them must be symmetric.

1. Introduction and main results

In this article, let Σ be a fixed C^3 compact convex hypersurface in $\mathbf{R}^{2n}$, i.e., Σ is the boundary of a compact and strictly convex region U in $\mathbf{R}^{2n}$. We denote the set of all such hypersurfaces by $\mathcal{H}(2n)$. Without loss of generality, we suppose U contains the origin. We denote the set of all compact convex hypersurfaces which are symmetric with respect to the origin by $\mathcal{SH}(2n)$, i.e., $\Sigma = -\Sigma$ for $\Sigma \in \mathcal{SH}(2n)$. We consider closed characteristics (τ, y) on Σ, which are solutions of the following problem

$$\begin{cases} \dot{y} = JN_\Sigma(y), \\ y(\tau) = y(0), \end{cases} \tag{1.1}$$

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where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, I_n is the identity matrix in $\mathbf{R}^n$, $\tau > 0$ and $N_\Sigma(y)$ is the outward normal vector of Σ at y normalized by the condition $N_\Sigma(y) \cdot y = 1$. Here $a \cdot b$ denotes the standard inner product of $a, b \in \mathbf{R}^{2n}$. A closed characteristic (τ, y) is *prime* if τ is the minimal period of y. Two closed characteristics (τ, y) and (σ, z) are *geometrically distinct* if $y(\mathbf{R}) \neq z(\mathbf{R})$. We denote by $\mathcal{J}(\Sigma)$ and $\tilde{\mathcal{J}}(\Sigma)$ the set of all closed characteristics (τ, y) on Σ with τ being the minimal period of y and the set of all geometrically distinct ones, respectively. Note that $\mathcal{J}(\Sigma) = \{\theta \cdot y \mid \theta \in S^1,\ y \text{ is prime}\}$, while $\tilde{\mathcal{J}}(\Sigma) = \mathcal{J}(\Sigma)/S^1$, where the natural S^1-action is defined by $\theta \cdot y(t) = y(t + \tau\theta)$, $\forall \theta \in S^1$, $t \in \mathbf{R}$. As defined in [7], a closed characteristic $(\tau, y) \in \mathcal{J}(\Sigma)$ on $\Sigma \in \mathcal{SH}(2n)$ is *symmetric* if $y(t) = -y(t + \frac{\tau}{2})$ for all $t \in \mathbf{R}$.

For the existence and multiplicity of geometrically distinct closed characteristics on convex compact hypersurfaces in $\mathbf{R}^{2n}$ we refer to [3,4,6,7,9–13,15] and references therein. Especially, for the symmetric hypersurfaces, the main theorem in [7] of C. Liu, Y. Long and C. Zhu implies $^{\#}\tilde{\mathcal{J}}(\Sigma) = n$ for any $\Sigma \in \mathcal{SH}(2n)$. Note that we have the following example of weakly non-resonant ellipsoid: Let $r = (r_1, \ldots, r_n)$ with $r_i > 0$ for $1 \leqslant i \leqslant n$. Define

$$\mathcal{E}_n(r) = \left\{ z = (x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbf{R}^{2n} \;\middle|\; \frac{1}{2} \sum_{i=1}^{n} \frac{x_i^2 + y_i^2}{r_i^2} = 1 \right\},$$

where $\frac{r_i}{r_j} \notin \mathbf{Q}$ whenever $i \neq j$. In this case, the corresponding Hamiltonian system is linear and all the solutions can be computed explicitly. Thus it is easy to verify that $^{\#}\tilde{\mathcal{J}}(\mathcal{E}_n(r)) = n$ and all of them are symmetric.

Motivated by these results, we prove the following result in this article:

Theorem 1.1. *Suppose $^{\#}\tilde{\mathcal{J}}(\Sigma) = n$ for some $\Sigma \in \mathcal{SH}(2n)$ and $n = 2$ or 3. Then any $(\tau, y) \in \mathcal{J}(\Sigma)$ is symmetric.*

The proof of the theorem is motivated by the methods in [9] and [13] by using the index iteration theory and the equivariant Morse theory for closed characteristics.

In this article, let $\mathbf{N}$, $\mathbf{N}_0$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ denote the sets of natural integers, non-negative integers, integers, rational numbers, real numbers, and complex numbers, respectively. Denote by $a \cdot b$ and $|a|$ the standard inner product and norm in $\mathbf{R}^{2n}$. Denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the standard L^2-inner product and L^2-norm. For an S^1-space X, we denote by X_{S^1} the homotopy quotient of X module the S^1-action, i.e., $X_{S^1} = S^\infty \times_{S^1} X$. We define the functions

$$\begin{cases} [a] = \max\{k \in \mathbf{Z} \mid k \leqslant a\}, \quad E(a) = \min\{k \in \mathbf{Z} \mid k \geqslant a\}, \\ \varphi(a) = E(a) - [a]. \end{cases} \tag{1.2}$$

Specially, $\varphi(a) = 0$ if $a \in \mathbf{Z}$, and $\varphi(a) = 1$ if $a \notin \mathbf{Z}$. In this article we use only $\mathbf{Q}$-coefficients for all homological modules.

2. Variational structure for closed characteristics

In the rest of this article, we fix a $\Sigma \in \mathcal{H}(2n)$ and assume the following condition on Σ:

(F) There exist only finitely many geometrically distinct closed characteristics $\{(\tau_j, y_j)\}_{1 \leqslant j \leqslant k}$ on Σ.

In this section, we review briefly the variational structure for closed characteristics.

Let $\hat{\tau} = \inf\{\tau_j \mid 1 \leqslant j \leqslant k\}$. Then by §2 of [13], for any $a > \hat{\tau}$, we can construct a function $\varphi_a \in C^\infty(\mathbf{R}, \mathbf{R}^+)$ which has 0 as its unique critical point in $[0, +\infty)$ such that φ_a is strictly convex for $t \geqslant 0$. Moreover, $\frac{\varphi_a'(t)}{t}$ is strictly decreasing for $t > 0$ together with $\lim_{t \to 0^+} \frac{\varphi_a'(t)}{t} = 1$ and $\varphi_a(0) = 0 = \varphi_a'(0)$.

More precisely, we define φ_a via Propositions 2.2 and 2.4 in [13]. Define the Hamiltonian function $H_a(x) = a\varphi_a(j(x))$ and consider the fixed period problem

$$\begin{cases} \dot{x}(t) = JH_a'\big(x(t)\big), \\ x(1) = x(0). \end{cases} \tag{2.1}$$

Then $H_a \in C^3(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R}) \cap C^1(\mathbf{R}^{2n}, \mathbf{R})$ is strictly convex. Solutions of (2.1) are $x \equiv 0$ and $x = \rho y(\tau t)$ with $\frac{\varphi_a'(\rho)}{\rho} = \frac{\tau}{a}$, where (τ, y) is a solution of (1.1). In particular, nonzero solutions of (2.1) are one to one correspondent to solutions of (1.1) with period $\tau < a$.

In the following, we use the Clarke–Ekeland dual action principle. As usual, let G_a be the Fenchel transform of H_a defined by $G_a(y) = \sup\{x \cdot y - H_a(x) \mid x \in \mathbf{R}^{2n}\}$. Then $G_a \in C^2(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R}) \cap C^1(\mathbf{R}^{2n}, \mathbf{R})$ is strictly convex. Let

$$L_0^2\big(S^1, \mathbf{R}^{2n}\big) = \left\{ u \in L^2\big([0,1], \mathbf{R}^{2n}\big) \;\middle|\; \int_0^1 u(t)\,dt = 0 \right\}. \tag{2.2}$$

Define a linear operator $M : L_0^2(S^1, \mathbf{R}^{2n}) \to L_0^2(S^1, \mathbf{R}^{2n})$ by $\frac{d}{dt} Mu(t) = u(t)$, $\int_0^1 Mu(t)\,dt = 0$. The dual action functional on $L_0^2(S^1, \mathbf{R}^{2n})$ is defined by

$$\Psi_a(u) = \int_0^1 \left(\frac{1}{2} Ju \cdot Mu + G_a(-Ju) \right) dt. \tag{2.3}$$

Then the functional $\Psi_a \in C^{1,1}(L_0^2(S^1, \mathbf{R}^{2n}), \mathbf{R})$ is bounded from below and satisfies the Palais–Smale condition. Suppose x is a solution of (2.1). Then $u = \dot{x}$ is a critical point of Ψ_a. Conversely, suppose u is a critical point of Ψ_a. Then there exists a unique $\xi \in \mathbf{R}^{2n}$ such that $Mu - \xi$ is a solution of (2.1). In particular, solutions of (2.1) are in one to one correspondence with critical points of Ψ_a. Moreover, $\Psi_a(u) < 0$ for every critical point $u \neq 0$ of Ψ_a.

Suppose u is a nonzero critical point of Ψ_a. Then the formal Hessian of Ψ_a at u is defined by

$$Q_a(v, v) = \int_0^1 \big(Jv \cdot Mv + G_a''(-Ju) Jv \cdot Jv \big)\,dt,$$

which defines an orthogonal splitting $L_0^2 = E_- \oplus E_0 \oplus E_+$ of $L_0^2(S^1, \mathbf{R}^{2n})$ into negative, zero and positive subspaces. The index of u is defined by $i(u) = \dim E_-$ and the nullity of u is defined by $\nu(u) = \dim E_0$. Let $u = \dot{x}$ be the critical point of Ψ_a such that x corresponds to the closed characteristic (τ, y) on Σ. Then the index $i(u)$ and the nullity $\nu(u)$ defined above coincide with the Ekeland indices defined by I. Ekeland in [1] and [2]. Specially $1 \leqslant \nu(u) \leqslant 2n - 1$ always holds.

We have a natural S^1-action on $L_0^2(S^1, \mathbf{R}^{2n})$ defined by $\theta \cdot u(t) = u(\theta + t)$ for all $\theta \in S^1$ and $t \in \mathbf{R}$. Clearly Ψ_a is S^1-invariant. For any $\kappa \in \mathbf{R}$, we denote by

$$\Lambda_a^\kappa = \big\{ u \in L_0^2\big(S^1, \mathbf{R}^{2n}\big) \;\big|\; \Psi_a(u) \leqslant \kappa \big\}. \tag{2.4}$$

For a critical point u of Ψ_a, we denote by

$$\Lambda_a(u) = \Lambda_a^{\Psi_a(u)} = \big\{ w \in L_0^2\big(S^1, \mathbf{R}^{2n}\big) \;\big|\; \Psi_a(w) \leqslant \Psi_a(u) \big\}. \tag{2.5}$$

Clearly, both sets are S^1-invariant. Since the S^1-action preserves Ψ_a, if u is a critical point of Ψ_a, then the whole orbit $S^1 \cdot u$ is formed by critical points of Ψ_a. Denote by $\operatorname{crit}(\Psi_a)$ the set of critical points of Ψ_a. Note that by the condition (F), the number of critical orbits of Ψ_a is finite. Hence as usual we can make the following definition.

Definition 2.1. Suppose u is a nonzero critical point of Ψ_a and $\mathcal{N}$ is an S^1-invariant open neighborhood of $S^1 \cdot u$ such that $\mathrm{crit}(\Psi_a) \cap (\Lambda_a(u) \cap \mathcal{N}) = S^1 \cdot u$. Then the S^1-critical modules of $S^1 \cdot u$ are defined by

$$C_{S^1,q}\big(\Psi_a, S^1 \cdot u\big) = H_q\big(\big(\Lambda_a(u) \cap \mathcal{N}\big)_{S^1}, \big(\big(\Lambda_a(u) \setminus S^1 \cdot u\big) \cap \mathcal{N}\big)_{S^1}\big).$$

We have the following proposition for critical modules.

Proposition 2.2. *(See Proposition 3.2 of [13].) The critical module $C_{S^1,q}(\Psi_a, S^1 \cdot u)$ is independent of a in the sense that if x_i are solutions of* (2.1) *with Hamiltonian functions $H_{a_i}(x) \equiv a_i \varphi_{a_i}(j(x))$ for $i = 1$ and 2 respectively such that both x_1 and x_2 correspond to the same closed characteristic (τ, y) on Σ. Then we have*

$$C_{S^1,q}\big(\Psi_{a_1}, S^1 \cdot \dot{x}_1\big) \cong C_{S^1,q}\big(\Psi_{a_2}, S^1 \cdot \dot{x}_2\big), \quad \forall q \in \mathbf{Z}.$$

Now let $u \neq 0$ be a critical point of Ψ_a with multiplicity $\mathrm{mul}(u) = m$, i.e., u corresponds to a closed characteristic $(m\tau, y) \subset \Sigma$ with (τ, y) being prime. Hence $u(t + \frac{1}{m}) = u(t)$ holds for all $t \in \mathbf{R}$. For any $p \in \mathbf{N}$, denote by u^p the unique critical point of Ψ_a corresponding to $(pm\tau, y)$. By §3 of [13], we can construct a $\mathbf{Z}_{pm}$-invariant local *characteristic manifold* $W(u^p)$ at u^p with dimension $\nu(u^p) - 1$ by using a finite-dimensional reduction and Gromoll–Meyer theory. Then we have the following proposition.

Proposition 2.3. *(See Proposition 3.10 of [13].) Let $u \neq 0$ be a critical point of Ψ_a with* $\mathrm{mul}(u) = 1$*. Then for all $p \in \mathbf{N}$ and $q \in \mathbf{Z}$, we have*

$$C_{S^1,q}\big(\Psi_a, S^1 \cdot u^p\big) \cong \big(H_{q-i(u^p)}\big(W\big(u^p\big) \cap \Lambda_a\big(u^p\big), \big(W\big(u^p\big) \setminus \big\{u^p\big\}\big) \cap \Lambda_a\big(u^p\big)\big)\big)^{\beta(u^p)\mathbf{Z}_p}, \tag{2.6}$$

where $\beta(u^p) = (-1)^{i(u^p) - i(u)}$. Thus

$$C_{S^1,q}\big(\Psi_a, S^1 \cdot u^p\big) = 0, \quad \text{for } q < i\big(u^p\big) \text{ or } q > i\big(u^p\big) + \nu\big(u^p\big) - 1. \tag{2.7}$$

We make the following definition.

Definition 2.4. Let $u \neq 0$ be a critical point of Ψ_a with $\mathrm{mul}(u) = 1$. Then for all $p \in \mathbf{N}$ and $l \in \mathbf{Z}$, let

$$\begin{aligned} k_{l,\pm 1}\big(u^p\big) &= \dim\big(H_l\big(W\big(u^p\big) \cap \Lambda_a\big(u^p\big), \big(W\big(u^p\big) \setminus \big\{u^p\big\}\big) \cap \Lambda_a\big(u^p\big)\big)\big)^{\pm \mathbf{Z}_p}, \\ k_l\big(u^p\big) &= \dim\big(H_l\big(W\big(u^p\big) \cap \Lambda_a\big(u^p\big), \big(W\big(u^p\big) \setminus \big\{u^p\big\}\big) \cap \Lambda_a\big(u^p\big)\big)\big)^{\beta(u^p)\mathbf{Z}_p}. \end{aligned}$$

$k_l(u^p)$s are called critical type numbers of u^p.

We have the following properties for critical type numbers.

Proposition 2.5. *(See Proposition 3.13 of [13].) Let $u \neq 0$ be a critical point of Ψ_a with* $\mathrm{mul}(u) = 1$*. Then there exists a minimal $K(u) \in 2\mathbf{N}$ such that*

$$\nu\big(u^{p+K(u)}\big) = \nu\big(u^p\big), \qquad i\big(u^{p+K(u)}\big) - i\big(u^p\big) \in 2\mathbf{Z},$$

and $k_l(u^{p+K(u)}) = k_l(u^p)$ for all $p \in \mathbf{N}$ and $l \in \mathbf{Z}$. We call $K(u)$ the minimal period of critical modules of iterations of the functional Ψ_a at u.

For a closed characteristic (τ, y) on Σ, we denote by $y^m \equiv (m\tau, y)$ the mth iteration of y for $m \in \mathbf{N}$. Let $a > \tau$ and choose φ_a as above. Determine ρ uniquely by $\frac{\varphi_a'(\rho)}{\rho} = \frac{\tau}{a}$. Let $x = \rho y(\tau t)$ and $u = \dot{x}$. Then we define the index $i(y^m)$ and nullity $\nu(y^m)$ of $(m\tau, y)$ for $m \in \mathbf{N}$ by

$$i(y^m) = i(u^m), \qquad \nu(y^m) = \nu(u^m).$$

These indices are independent of a when a tends to infinity. Now the mean index of (τ, y) is defined by

$$\hat{i}(y) = \lim_{m\to\infty} \frac{i(y^m)}{m}. \tag{2.8}$$

By Proposition 2.2, we define the critical type numbers $k_l(y^m)$ of y^m to be $k_l(u^m)$, where u^m is the critical point of Ψ_a corresponding to y^m. We also define $K(y) = K(u)$. Then we have

Proposition 2.6. *(See Proposition 2.6 of [14].) We have $k_l(y^m) = 0$ for $l \notin [0, \nu(y^m) - 1]$ and it can take only values 0 or 1 when $l = 0$ or $l = \nu(y^m) - 1$. Moreover, the following properties hold:*

(i) $k_0(y^m) = 1$ *implies* $k_l(y^m) = 0$ *for* $1 \leqslant l \leqslant \nu(y^m) - 1$.
(ii) $k_{\nu(y^m)-1}(y^m) = 1$ *implies* $k_l(y^m) = 0$ *for* $0 \leqslant l \leqslant \nu(y^m) - 2$.
(iii) $k_l(y^m) \geqslant 1$ *for some* $1 \leqslant l \leqslant \nu(y^m) - 2$ *implies* $k_0(y^m) = k_{\nu(y^m)-1}(y^m) = 0$.
(iv) *If* $\nu(y^m) \leqslant 3$, *then at most one of the* $k_l(y^m)$*s for* $0 \leqslant l \leqslant \nu(y^m) - 1$ *can be nonzero.*

For a closed characteristic (τ, y) on Σ, we define

$$\hat{\chi}(y) = \frac{1}{K(y)} \sum_{\substack{1\leqslant m\leqslant K(y)\\ 0\leqslant l\leqslant 2n-2}} (-1)^{i(y^m)+l} k_l(y^m). \tag{2.9}$$

We have the following mean index identity for closed characteristics.

Theorem 2.7. *(See Theorem 1.2 of [13].) Suppose $\Sigma \in \mathcal{H}(2n)$ satisfies $^{\#}\tilde{\mathcal{J}}(\Sigma) < +\infty$. Denote all the geometrically distinct closed characteristics by $\{(\tau_j, y_j)\}_{1\leqslant j\leqslant k}$. Then the following identity holds*

$$\sum_{1\leqslant j\leqslant k} \frac{\hat{\chi}(y_j)}{\hat{i}(y_j)} = \frac{1}{2}.$$

Recall that for a principal $U(1)$-bundle $E \to B$, the Fadell–Rabinowitz index (cf. [5]) of E is defined to be $\sup\{k \mid c_1(E)^{k-1} \neq 0\}$, where $c_1(E) \in H^2(B, \mathbf{Q})$ is the first rational Chern class. For a $U(1)$-space, i.e., a topological space X with a $U(1)$-action, the Fadell–Rabinowitz index is defined to be the index of the bundle $X \times S^\infty \to X \times_{U(1)} S^\infty$, where $S^\infty \to CP^\infty$ is the universal $U(1)$-bundle.

As in [2, p. 199], choose some $\alpha \in (1, 2)$ and associate with U a convex function H such that $H(\lambda x) = \lambda^\alpha H(x)$ for $\lambda \geqslant 0$. Consider the fixed period problem

$$\begin{cases} \dot{x}(t) = JH'(x(t)), \\ x(1) = x(0). \end{cases} \tag{2.10}$$

Define

$$L_0^{\frac{\alpha}{\alpha-1}}(S^1, \mathbf{R}^{2n}) = \left\{ u \in L^{\frac{\alpha}{\alpha-1}}(S^1, \mathbf{R}^{2n}) \;\middle|\; \int_0^1 u\,dt = 0 \right\}. \tag{2.11}$$

The corresponding Clarke–Ekeland dual action functional is defined by

$$\Phi(u)=\int_0^1\left(\frac{1}{2}Ju\cdot Mu+H^*(-Ju)\right)dt,\quad \forall u\in L_0^{\frac{\alpha}{\alpha-1}}\left(S^1,\mathbf{R}^{2n}\right),\tag{2.12}$$

where Mu is defined as above and H^* is the Fenchel transform of H.

For any $\kappa\in\mathbf{R}$, we denote by

$$\Phi^{\kappa-}=\left\{u\in L_0^{\frac{\alpha}{\alpha-1}}\left(S^1,\mathbf{R}^{2n}\right)\,\middle|\,\Phi(u)<\kappa\right\}.\tag{2.13}$$

Then as in [2, p. 218], we define

$$c_i=\inf\left\{\delta\in\mathbf{R}\,\middle|\,\hat{I}\left(\Phi^{\delta-}\right)\geqslant i\right\},\tag{2.14}$$

where $\hat{I}$ is the Fadell–Rabinowitz index given above. Then by Proposition 3 in p. 218 of [2], we have

Proposition 2.8. *Every c_i is a critical value of Φ. If $c_i=c_j$ for some $i<j$, then there are infinitely many geometrically distinct closed characteristics on Σ.*

As in Definition 2.1, we define the following.

Definition 2.9. Suppose u is a nonzero critical point of Φ, and $\mathcal{N}$ is an S^1-invariant open neighborhood of $S^1\cdot u$ such that $\mathrm{crit}(\Phi)\cap(\Lambda(u)\cap\mathcal{N})=S^1\cdot u$. Then the S^1-critical modules of $S^1\cdot u$ are defined by

$$C_{S^1,q}\left(\Phi,S^1\cdot u\right)=H_q\left(\left(\Lambda(u)\cap\mathcal{N}\right)_{S^1},\left(\left(\Lambda(u)\setminus S^1\cdot u\right)\cap\mathcal{N}\right)_{S^1}\right),\tag{2.15}$$

where $\Lambda(u)=\{w\in L_0^{\frac{\alpha}{\alpha-1}}(S^1,\mathbf{R}^{2n})\mid\Phi(w)\leqslant\Phi(u)\}$.

Comparing with Theorem 4 in p. 219 of [2], we have the following.

Proposition 2.10. *(See Proposition 3.5 of [14].) For every $i\in\mathbf{N}$, there exists a point $u\in L_0^{\frac{\alpha}{\alpha-1}}(S^1,\mathbf{R}^{2n})$ such that*

$$\Phi'(u)=0,\qquad\Phi(u)=c_i,\tag{2.16}$$

$$C_{S^1,2(i-1)}\left(\Phi,S^1\cdot u\right)\neq 0.\tag{2.17}$$

Proposition 2.11. *(See Proposition 3.6 of [14].) Suppose u is the critical point of Φ found in Proposition 2.10. Then we have*

$$C_{S^1,2(i-1)}\left(\Psi_a,S^1\cdot u_a\right)\cong C_{S^1,2(i-1)}\left(\Phi,S^1\cdot u\right)\neq 0,\tag{2.18}$$

where Ψ_a is given by (2.3) and $u_a\in L_0^2(S^1,\mathbf{R}^{2n})$ is its critical point corresponding to u in the natural sense.

3. Proof of the main theorem

In this section, we give the proof of the main theorem.

Theorem 3.1. *(Cf. Theorem 15.1.1 of [8].) Suppose $(\tau, y) \in \mathcal{J}(\Sigma)$. Then we have*

$$i(y^m) \equiv i(m\tau, y) = i(y, m) - n, \qquad \nu(y^m) \equiv \nu(m\tau, y) = \nu(y, m), \quad \forall m \in \mathbf{N}, \tag{3.1}$$

where $i(y, m)$ and $\nu(y, m)$ are the Maslov-type index and nullity of $(m\tau, y)$ defined by Conley, Zehnder and Long (cf. §5.4 of [8]).

In the rest of this article, we fix a $\Sigma \in \mathcal{SH}(2n)$.

We have the following property, cf. Lemma 4.2 of [7].

Lemma 3.2. *Suppose $(\tau, y) \in \mathcal{J}(\Sigma)$, then $(\tau, -y) \in \mathcal{J}(\Sigma)$ and either $\mathcal{O}(y) = \mathcal{O}(-y)$ or $\mathcal{O}(y) \cap \mathcal{O}(-y) = \emptyset$, where $\mathcal{O}(\pm y) = \{\pm y(t) \mid t \in \mathbf{R}\}$. Moreover, if $\mathcal{O}(y) \cap \mathcal{O}(-y) \neq \emptyset$, then we have*

$$y(t) = -y\left(t + \frac{\tau}{2}\right), \quad \forall t \in \mathbf{R}.$$

As in [7], we call a closed characteristic (τ, y) on $\Sigma \in \mathcal{SH}(2n)$ *symmetric* if $\mathcal{O}(y) \cap \mathcal{O}(-y) \neq \emptyset$, *non-symmetric* if $\mathcal{O}(y) \cap \mathcal{O}(-y) = \emptyset$. Thus if (τ, y) is non-symmetric, then (τ, y) and $(\tau, -y)$ are geometrically distinct; if (τ, y) is symmetric, then (τ, y) and $(\tau, -y)$ are geometrically the same.

Lemma 3.3. *Suppose $\Sigma \in \mathcal{SH}(2n)$ and $(\tau, y) \in \mathcal{J}(\Sigma)$. Then we have*

$$\big(i(y^m), \nu(y^m)\big) = \big(i((-y)^m), \nu((-y)^m)\big), \qquad \Phi(u^m) = \Phi\big((-u)^m\big), \quad \forall m \in \mathbf{N}, \tag{3.2}$$

$$C_{S^1,q}\big(\Psi_a, S^1 \cdot u^m\big) \cong C_{S^1,q}\big(\Psi_a, S^1 \cdot (-u)^m\big), \quad \forall m \in \mathbf{N}, \ \forall q \in \mathbf{Z}, \tag{3.3}$$

where we denote simply by $(\pm u)^m$ the critical point of Φ or Ψ_a corresponding to $(\pm y)^m$.

Proof. Note that (3.2) was proved in [7]. Hence we only need to prove (3.3). Since $\Sigma = -\Sigma$, we have $H_a(x) = H_a(-x)$. Thus we have a natural $\mathbf{Z}_2$-action on $L_0^2(S^1, \mathbf{R}^{2n})$ defined by $v \mapsto -v$ and the functional Ψ_a defined in (2.3) is $\mathbf{Z}_2$-invariant. Hence (3.3) holds. □

Proof of Theorem 1.1. By the assumption (F) at the beginning of Section 2, we denote by $\{(\tau_j, y_j)\}_{1 \leqslant j \leqslant k}$ all the geometrically distinct closed characteristics on Σ, and by $\gamma_j \equiv \gamma_{y_j}$ the associated symplectic path of (τ_j, y_j) on Σ for $1 \leqslant j \leqslant k$. Then by Lemma 15.2.4 of [8], there exist $P_j \in \mathrm{Sp}(2n)$ and $M_j \in \mathrm{Sp}(2n-2)$ such that

$$\gamma_j(\tau_j) = P_j^{-1}\big(N_1(1,1) \diamond M_j\big)P_j, \quad \forall 1 \leqslant j \leqslant k, \tag{3.4}$$

where $N_1(1, b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ for $b \in \mathbf{R}$.

Firstly we consider the case $n = 2$. Suppose there are exactly two geometrically distinct closed characteristics (τ_1, y_1) and (τ_2, y_2) on $\Sigma \in \mathcal{SH}(4)$. If (τ_1, y_1) is non-symmetric, we must have $(\tau_2, y_2) = (\tau_1, -y_1)$ by Lemma 3.2 since otherwise we would have at least 3 geometrically distinct closed characteristics on Σ. Thus by Lemma 3.3, (2.8) and (2.9), we have

$$\hat{\chi}(y_2) = \hat{\chi}(-y_1) = \hat{\chi}(y_1), \qquad \hat{i}(y_2) = \hat{i}(-y_1) = \hat{i}(y_1). \tag{3.5}$$

Hence by Theorem 2.7, the following identity holds

$$\frac{1}{2}=\frac{\hat{\chi}(y_1)}{\hat{i}(y_1)}+\frac{\hat{\chi}(y_2)}{\hat{i}(y_2)}=\frac{2\hat{\chi}(y_1)}{\hat{i}(y_1)}. \tag{3.6}$$

This implies $\hat{i}(y_1)=\hat{i}(y_2)\in\mathbf{Q}$. On the other hand, by Theorem 1.4 of [13], we have $\hat{i}(y_1)\notin\mathbf{Q}$ and $\hat{i}(y_2)\notin\mathbf{Q}$. This contradiction proves Theorem 1.1 for the case $n=2$.

Next we consider the case $n=3$. Suppose there are exactly three geometrically distinct closed characteristics (τ_1,y_1), (τ_2,y_2) and (τ_3,y_3) on $\Sigma\in\mathcal{SH}(6)$. If (τ_1,y_1) is non-symmetric, by Lemma 3.2, we may assume $(\tau_2,y_2)=(\tau_1,-y_1)$ and (τ_3,y_3) is symmetric, since otherwise we would have at least 4 geometrically distinct closed characteristics on Σ. Thus by (3.5) and Theorem 2.7 we have

$$\frac{1}{2}=\frac{\hat{\chi}(y_1)}{\hat{i}(y_1)}+\frac{\hat{\chi}(y_2)}{\hat{i}(y_2)}+\frac{\hat{\chi}(y_3)}{\hat{i}(y_3)}=\frac{2\hat{\chi}(y_1)}{\hat{i}(y_1)}+\frac{\hat{\chi}(y_3)}{\hat{i}(y_3)}. \tag{3.7}$$

By Corollary 15.1.4 of [8], we have $i(y_j,1)\geqslant 3$ for $1\leqslant j\leqslant 3$. Note that $e(\gamma_j(\tau_j))\leqslant 6$ for $1\leqslant j\leqslant k$, where $e(M)$ is the total algebraic multiplicity of all eigenvalues of M on the unit circle $\mathbf{U}=\{z\in\mathbf{C}\mid |z|=1\}$ in the complex plane $\mathbf{C}$. Hence Theorem 10.2.4 of [8] yields

$$\begin{aligned} i(y_j,m)+\nu(y_j,m)&\leqslant i(y_j,m+1)-i(y_j,1)+\frac{e(\gamma_j(\tau_j))}{2}-1\\ &\leqslant i(y_j,m+1)-1,\quad \forall m\in\mathbf{N},\ 1\leqslant j\leqslant 3. \end{aligned}\tag{3.8}$$

Using the common index jump theorem (Theorems 4.3 and 4.4 of [9], Theorems 11.2.1 and 11.2.2 of [8]), we obtain some $(T,m_1,\dots,m_3)\in\mathbf{N}^4$ such that the following hold by (11.2.6), (11.2.7) and (11.2.26) of [8]:

$$i(y_j,2m_j)\geqslant 2T-\frac{e(\gamma_j(\tau_j))}{2}, \tag{3.9}$$

$$i(y_j,2m_j)+\nu(y_j,2m_j)\leqslant 2T+\frac{e(\gamma_j(\tau_j))}{2}-1, \tag{3.10}$$

$$i(y_j,2m_j+m)\geqslant 2T+i(y_j,1),\quad \forall m\geqslant 1. \tag{3.11}$$

$$i(y_j,2m_j-1)+\nu(y_j,2m_j-1)=2T-\big(i(y_j,1)+2S^+_{\gamma_j(\tau_j)}(1)-\nu(y_j,1)\big), \tag{3.12}$$

$$i(y_j,2m_j-m)+\nu(y_j,2m_j-m)\leqslant 2T-i(y_j,1)-1,\quad \forall m\geqslant 2, \tag{3.13}$$

where $S^+_M(1)$ is the splitting number defined in §9 of [8].

By [8, p. 340], we have

$$\begin{aligned} 2S^+_{\gamma_j(\tau_j)}(1)-\nu(y_j,1)&=2S^+_{N_1(1,1)}(1)-\nu_1\big(N_1(1,1)\big)+2S^+_{M_j}(1)-\nu_1(M_j)\\ &=1+2S^+_{M_j}(1)-\nu_1(M_j)\\ &\geqslant -1,\quad 1\leqslant j\leqslant 3. \end{aligned}\tag{3.14}$$

In the last inequality, we have used the fact that the worst case for $2S^+_{M_j}(1)-\nu_1(M_j)$ happens if and only if $M_j=N_1(1,-1)^{\diamond 2}$ which gives the lower bound -2.

By Theorem 1.1 of [14], there exist at least two geometrically distinct closed characteristics possessing irrational mean indices on any $\Sigma\in\mathcal{H}(6)$ under the assumption (F). Thus we have the following two cases:

Case 1. We have $\hat{i}(y_1) = \hat{i}(y_2) \notin \mathbf{Q}$ and $\hat{i}(y_3) \notin \mathbf{Q}$.

In this case, by Theorem 8.3.1, Corollary 8.3.2 of [8] and Theorem 3.1, $\hat{i}(y_j) \notin \mathbf{Q}$ implies $M_j \in \mathrm{Sp}(4)$ in (3.4) can be connected to $R(\theta_j) \diamond Q_j$ within $\Omega^0(M_j)$ for some $\frac{\theta_j}{\pi} \notin \mathbf{Q}$ and $Q_j \in \mathrm{Sp}(2)$ for $1 \leqslant j \leqslant 3$, where $R(\theta) = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$ for $\theta \in \mathbf{R}$. Here we use notations from Definition 1.8.5 and Theorem 1.8.10 of [8]. Hence we have

$$2S^{+}_{\gamma_j(\tau_j)}(1) - \nu(y_j, 1) \geqslant 0, \quad 1 \leqslant j \leqslant 3. \tag{3.15}$$

Thus (3.9)–(3.13) becomes

$$i(y_j, 2m_j) \geqslant 2T - 3, \tag{3.16}$$

$$i(y_j, 2m_j) + \nu(y_j, 2m_j) - 1 \leqslant 2T + 1, \tag{3.17}$$

$$i(y_j, 2m_j + m) \geqslant 2T + 3, \quad \forall m \geqslant 1, \tag{3.18}$$

$$i(y_j, 2m_j - m) + \nu(y_j, 2m_j - m) - 1 \leqslant 2T - 4, \quad \forall m \geqslant 1. \tag{3.19}$$

Note that by Theorem 3.1

$$i(y_j^m) = i(y_j, m) - 3, \quad \forall m \in \mathbf{N},\ 1 \leqslant j \leqslant 3. \tag{3.20}$$

Hence we have

$$C_{S^1, 2T-2l}(\Psi_a, S^1 \cdot u_j^m) = 0, \quad \forall m \neq 2m_j,\ 1 \leqslant j \leqslant 3,\ 1 \leqslant l \leqslant 3. \tag{3.21}$$

In fact, by (3.18) and (3.20), we have $i(u_j^m) = i(y_j^m) \geqslant 2T$ for all $m > 2m_j$ and $i(u_j^m) + \nu(u_j^m) - 1 = i(y_j^m) + \nu(y_j^m) - 1 \leqslant 2T - 7$ for all $m < 2m_j$. Thus (3.21) holds by Proposition 2.6.

By Propositions 2.10 and 2.11 we can find $p, q, r \in \{1, 2, 3\}$ such that

$$\Phi'(u_p^{2m_p}) = 0, \qquad \Phi(u_p^{2m_p}) = c_{T-2}, \qquad C_{S^1, 2T-6}(\Psi_a, S^1 \cdot u_p^{2m_p}) \neq 0, \tag{3.22}$$

$$\Phi'(u_q^{2m_q}) = 0, \qquad \Phi(u_q^{2m_q}) = c_{T-1}, \qquad C_{S^1, 2T-4}(\Psi_a, S^1 \cdot u_q^{2m_q}) \neq 0, \tag{3.23}$$

$$\Phi'(u_r^{2m_r}) = 0, \qquad \Phi(u_r^{2m_r}) = c_{T}, \qquad C_{S^1, 2T-2}(\Psi_a, S^1 \cdot u_r^{2m_r}) \neq 0, \tag{3.24}$$

where we denote also by $u_p^{2m_p}$, $u_q^{2m_q}$ and $u_r^{2m_r}$ the corresponding critical points of Φ and which will not be confused. By Proposition 2.8 we have $c_{T-2} < c_{T-1} < c_T$. Hence p, q, r are pairwise distinct. On the other hand, by Lemma 3.3 we have $\Phi(u_1^{2m_1}) = \Phi(u_2^{2m_2})$, where we use $m_1 = m_2$ which follows from Theorems 4.3 of [9]. This contradiction proves the theorem in Case 1.

Case 2. We have $\hat{i}(y_1) = \hat{i}(y_2) \notin \mathbf{Q}$ and $\hat{i}(y_3) \in \mathbf{Q}$.

In this case, by (3.7) we have $\hat{\chi}(y_1) = \hat{\chi}(y_2) = 0$. Note that if $M_3 = N_1(1, -1)^{\diamond 2}$ and $i(y_3, 1) = 3$ does not hold, we still have (3.16)–(3.19), thus the proof of Case 1 remains valid and derives a contradiction. Hence we only need to consider the case $M_3 = N_1(1, -1)^{\diamond 2}$ and $i(y_3, 1) = 3$. By Theorem 8.3.1 of [8] and Theorem 3.1, we obtain

$$i(y_3^m) = m\big(i(y_3, 1) + 1\big) - 1 - 3 = 4m - 4, \qquad \nu(y_3^m) = 3, \quad \forall m \in \mathbf{N}. \tag{3.25}$$

By Proposition 3.13 of [13], we have $K(y_3) = 2$. Thus by Proposition 2.6, we have

$$\begin{aligned}\hat{\chi}(y_3) &= \frac{1}{K(y_3)} \sum_{\substack{1\leqslant m\leqslant 2\\ 0\leqslant l\leqslant 2}} (-1)^{i(y_3^m)+l} k_l\big(y_3^m\big)\\ &= \frac{1}{2}\big(k_0(y_3) - k_1(y_3) + k_2(y_3) + k_0\big(y_3^2\big) - k_1\big(y_3^2\big) + k_2\big(y_3^2\big)\big)\\ &\leqslant 1.\end{aligned} \tag{3.26}$$

Now (3.7), (3.25) and (3.26) yield a contradiction:

$$\frac{1}{2} = \frac{\hat{\chi}(y_3)}{\hat{i}(y_3)} \leqslant \frac{1}{4},$$

which proves Theorem 1.1 in Case 2.

The proof of Theorem 1.1 is complete. □

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